

SOME WEIGHTED OSTROWSKI TYPE INEQUALITIES ON TIME SCALES INVOLVING COMBINATION OF WEIGHTED Δ -INTEGRAL MEANS

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ABSTRACT. In this paper we obtain some weighted generalizations of Ostrowski type inequalities on time scales involving combination of weighted Δ -integral means, i.e., a weighted Ostrowski type inequality on time scales involving combination of weighted Δ -integral means, two weighted Ostrowski type inequalities for two functions on time scales, and some weighted perturbed Ostrowski type inequalities on time scales. We also give some other interesting inequalities and recapture some known results as special cases.

1. INTRODUCTION

In 1937, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [28]. The result is nowadays known as the Ostrowski inequality [2, 12, 13, 35], which can be obtained by using the Montgomery identity. Recently, Ahmad et. al [2] developed some new Ostrowski inequalities involving two functions, by using an identity of Dragomir and Barnett proved in [11]. In [35], Tseng, Hwang and Dragomir established the following generalizations of weighted Ostrowski type inequalities for mappings of bounded variation.

Theorem A. *Let us have $0 \leq \alpha \leq 1$, $g : [a, b] \rightarrow [0, \infty)$ continuous and positive on (a, b) and let $h : [a, b] \rightarrow \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Let $c = h^{-1}((1 - \frac{\alpha}{2})h(a) + \frac{\alpha}{2}h(b))$ and $d = h^{-1}(\frac{\alpha}{2}h(a) + (1 - \frac{\alpha}{2})h(b))$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation. Then, for all $x \in [c, d]$, we have*

$$(1.1) \quad \left| \int_a^b f(t)g(t)dt - \left[(1 - \alpha)f(x) + \alpha \frac{f(a) + f(b)}{2} \right] \int_a^b g(t)dt \right| \leq K \bigvee_a^b(f),$$

where

$$(1.2) \quad K := \begin{cases} \frac{1 - \alpha}{2} \int_a^b g(t)dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right|, & 0 \leq \alpha \leq \frac{1}{2}, \\ \max \left\{ \frac{1 - \alpha}{2} \int_a^b g(t)dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right|, \frac{\alpha}{2} \int_a^b g(t)dt \right\}, & \frac{1}{2} < \alpha < \frac{2}{3}, \\ \frac{\alpha}{2} \int_a^b g(t)dt, & \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

and $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. In (3.1), the constant $\frac{1-\alpha}{2}$ for $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

In 1988, Hilger introduced the time scale theory in order to unify continuous and discrete analysis [14]. Such theory has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in population dynamics [4], economics [3], physics [34], space weather [19] and so on. Recently, many authors studied the theory of certain integral inequalities on time scales (see [7, 8, 10, 15, 20, 21, 22, 23, 24, 25, 27, 32, 33, 36, 37]). The Ostrowski inequality and the Montgomery identity were generalized by Bohner and Matthews to an arbitrary time scale [8], unifying the discrete, the continuous, and the quantum cases:

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Theorem B (Ostrowski's inequality on time scales [8]). *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then*

$$(1.3) \quad \left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s \right| \leq \frac{M}{b-a} [h_2(t, a) + h_2(t, b)],$$

where $h_2(\cdot, \cdot)$ is defined by Definition 5 below and $M = \sup_{a < t < b} |f^\Delta(t)| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.3) cannot be replaced by a smaller one.

Very recently, the authors [16] gave some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ -integral means by using the kernel given in [9]. The purpose of this paper is to obtain some weighted Ostrowski type inequalities on time scales involving combination of weighted Δ -integral means. We first establish a weighted Ostrowski type inequality on time scales involving combination of weighted Δ -integral means. Then we derive two weighted Ostrowski type inequalities for two functions on time scales. Finally, four weighted perturbed Ostrowski type inequalities on time scales are established. We also give some other interesting inequalities and recapture some known results as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. Some weighted Ostrowski type inequalities on time scales are derived in Section 3.

2. TIME SCALES ESSENTIALS

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [14], the books [5, 6, 18], and the survey [1].

Definition 1. A time scale \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} . For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$. The jump operators σ and ρ allow the classification of points in \mathbb{T} as follows. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, the t is called right-dense, and if $\rho(t) = t$ then t is called left-dense, Points that both right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the graininess function. The set \mathbb{T}^k is defined as follows: if \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2. Let $f : \mathbb{T} \rightarrow \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

Theorem C. Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

Definition 3. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [5, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 4. Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $F : \mathbb{T} \rightarrow \mathbb{R}$ is called the antiderivative of f on \mathbb{T} if it satisfies $F^\Delta(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ -integral of f as

$$\int_a^t f(s) \Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

Theorem D. Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then

- (1) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$,
- (2) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t$,
- (3) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$,
- (4) $\int_a^b f(t) g^\Delta(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(\sigma(t)) \Delta t$,

Theorem E. If f is Δ -integrable on $[a, b]$, then so is $|f|$, and

$$\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b |f(t)| \Delta t.$$

Definition 5. Let $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t, s) = 1 \quad \text{for all } s, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau \quad \text{for all } s, t \in \mathbb{T}.$$

3. MAIN RESULTS

3.1. A weighted Ostrowski type inequality on time scales. We first establish a weighted Ostrowski type inequality on time scales involving combination of weighted Δ -integral means. For this purpose, we need the following lemma:

Lemma 1. Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f, h : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have the weighted Montgomery identity on time scales involving combination of weighted Δ -integral means

$$\begin{aligned} \int_a^b P(x, t) f^\Delta(t) \Delta t &= \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\ &\quad - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right], \end{aligned} \quad (3.1)$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero,

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b, \end{cases} \quad (3.2)$$

which is the weighted version of the kernel given in [9].

Proof. Using Theorem D (4), we have

$$\begin{aligned} &\int_a^x \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right) f^\Delta(t) \Delta t \\ &= \frac{\alpha}{\alpha + \beta} \left(\frac{h(x) - h(a)}{x - a} \right) f(x) - \frac{\alpha}{(\alpha + \beta)(x - a)} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} &\int_x^b \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right) f^\Delta(t) \Delta t \\ &= \frac{\beta}{\alpha + \beta} \left(\frac{h(b) - h(x)}{b - x} \right) f(x) - \frac{\beta}{(\alpha + \beta)(b - x)} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t. \end{aligned} \quad (3.4)$$

Therefore, the identity (3.1) is proved by adding the above two identities. \square

Remark 1. If we take $h(t) = t$ and $\mathbb{T} = \mathbb{R}$ in Lemma 1, we obtain the identity given in [9, Lemma 1].

Corollary 1. *If we take $\mathbb{T} = \mathbb{R}$ in Lemma 1, then we get the weighted Montgomery identity*

$$\begin{aligned} \int_a^b P(x, t) f'(t) dt &= \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\ &\quad - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h'(t) f(t) dt + \frac{\beta}{b - x} \int_x^b h'(t) f(t) dt \right], \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Corollary 2. *If we take $\mathbb{T} = \mathbb{Z}$ in Lemma 1, then we get*

$$\begin{aligned} \sum_{t=a}^{b-1} P(x, t) \Delta f(t) &= \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\ &\quad - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1) \Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1) \Delta h(t) \right], \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x - 1, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b - 1. \end{cases}$$

Corollary 3. *If we take $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Lemma 1, then we get*

$$\begin{aligned} \int_a^b P(x, t) D_q f(t) d_q t &= \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\ &\quad - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right], \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Here, for $s, t \in q^{\mathbb{Z}} \cup \{0\}$ with $t \geq s$, we use the definitions

$$(D_q f)(t) = \frac{f(qt) - f(t)}{(q - 1)t} \quad \text{and} \quad \int_s^t f(\eta) d_q \eta = (q - 1) \sum_{\ell=\log_q(s)}^{\log_q(t/q)} f(q^\ell) q^\ell,$$

by adopting the convention that $\log_q(0) := -\infty$ and $\log_q(\infty) := \infty$ (see [17]).

Theorem 1. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f, h : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &\left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\ &\quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right| \\ &\leq \frac{M}{\alpha + \beta} \int_a^b |P(x, t)| \Delta t, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero,

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M = \sup_{a < t < b} |f^\Delta(t)| < \infty.$$

Proof. The proof of Theorem 1 can be done easily from Lemma 1 by using the properties of modulus and the definition of $h_2(\cdot, \cdot)$. \square

Remark 2. In the case of $\alpha = x - a$ and $\beta = b - x$ in Theorem 1, we get

$$\left| \frac{h(b) - h(a)}{b - a} f(x) - \frac{1}{b - a} \int_a^b h^\Delta(t) f(\sigma(t)) \Delta t \right| \leq \frac{M}{b - a} \left[\int_a^x |h(t) - h(a)| \Delta t + \int_x^b |h(b) - h(t)| \Delta t \right],$$

which is the weighted version of (1.3).

Corollary 4. In the case of $\mathbb{T} = \mathbb{R}$ in Theorem 1, we have

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h'(t) f(t) dt + \frac{\beta}{b - x} \int_x^b h'(t) f(t) dt \right] \right| \\ & \leq \frac{M}{\alpha + \beta} \int_a^b |P(x, t)| dt, \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M = \sup_{a < t < b} |f'(t)| < \infty.$$

Corollary 5. In the case of $\mathbb{T} = \mathbb{Z}$ in Theorem 1, we have

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1) \Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1) \Delta h(t) \right] \right| \\ & \leq \frac{M}{\alpha + \beta} \sum_{t=a}^{b-1} |P(x, t)|, \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x - 1, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b - 1. \end{cases}$$

and

$$M = \sup_{a < t < b} |\Delta f(t)| < \infty.$$

Corollary 6. In the case of $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 1, we have

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right] \right| \\ & \leq \frac{M}{\alpha + \beta} \int_a^b |P(x, t)| d_q t, \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M = \sup_{a < t < b} |(D_q f)(t)| < \infty.$$

3.2. Weighted Ostrowski type inequalities for two functions on time scales. Then, we derive two weighted Ostrowski type inequalities for two functions on time scales.

Theorem 2. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f, g, h : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have*

$$\begin{aligned}
 & \left| \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\
 & \quad - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right. \\
 & \quad \left. \left. + f(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \right\} \right| \\
 (3.5) \quad & \leq \frac{M_1 |g(x)| + M_2 |f(x)|}{2(\alpha + \beta)} \left[\int_a^b |P(x, t)| \Delta t \right]
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| f(x)g(x) \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right]^2 - \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\
 & \quad \times \left\{ f(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \right. \\
 & \quad \left. + g(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right\} \\
 & \quad + \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \\
 & \quad \times \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \Big| \\
 (3.6) \quad & \leq (\alpha + \beta)^2 \left(\int_a^b |P(x, t)| \Delta t \right)^2,
 \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero,

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M_1 = \sup_{a < t < b} |f^\Delta(t)| < \infty \quad \text{and} \quad M_2 = \sup_{a < t < b} |g^\Delta(t)| < \infty.$$

Proof. We have

$$\begin{aligned}
 & \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\
 & \quad - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \\
 (3.7) \quad & = \int_a^b P(x, t) f^\Delta(t) \Delta t
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\
 & - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \\
 (3.8) \quad & = \int_a^b P(x, t) g^\Delta(t) \Delta t.
 \end{aligned}$$

Multiplying (3.7) by $g(x)$ and (3.8) by $f(x)$, adding the resultant identities, we have

$$\begin{aligned}
 & \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\
 & - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right. \\
 & \left. + f(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \right\} \\
 & = \frac{1}{2} \left[g(x) \int_a^b P(x, t) f^\Delta(t) \Delta t + f(x) \int_a^b P(x, t) g^\Delta(t) \Delta t \right].
 \end{aligned}$$

Using the properties of modulus, we get

$$\begin{aligned}
 & \left| \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\
 & - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right. \\
 & \left. \left. + f(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \right\} \right| \\
 & \leq \frac{1}{2} \left[|g(x)| \int_a^b |P(x, t)| |f^\Delta(t)| \Delta t + |f(x)| \int_a^b |P(x, t)| |g^\Delta(t)| \Delta t \right] \\
 & \leq \frac{M_1 |g(x)| + M_2 |f(x)|}{2(\alpha + \beta)} \left[\int_a^b |P(x, t)| \Delta t \right].
 \end{aligned}$$

This completes the proof of the inequality (3.5).

Multiplying the left sides and right sides of (3.7) and (3.8), we get

$$\begin{aligned}
 & f(x)g(x) \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right]^2 \\
 & - \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \left\{ f(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \right. \\
 & \left. + g(x) \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right\} \\
 & + \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \\
 & \times \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) g(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) g(\sigma(t)) \Delta t \right] \\
 & = (\alpha + \beta)^2 \left(\int_a^b P(x, t) f^\Delta(t) \Delta t \right) \left(\int_a^b P(x, t) g^\Delta(t) \Delta t \right).
 \end{aligned}$$

Using the properties of modulus, we can easily obtain (3.6). \square

Corollary 7. *In the case of $\mathbb{T} = \mathbb{R}$ in Theorem 2, we have*

$$\begin{aligned} & \left| \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\ & - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \int_a^x h'(t)f(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)f(t)dt \right] \right. \\ & \left. \left. + f(x) \left[\frac{\alpha}{x - a} \int_a^x h'(t)g(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)g(t)dt \right] \right\} \right| \\ & \leq \frac{M_1 |g(x)| + M_2 |f(x)|}{2(\alpha + \beta)} \left[\int_a^b |P(x, t)| \Delta t \right] \end{aligned}$$

and

$$\begin{aligned} & \left| f(x)g(x) \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right]^2 \right. \\ & - \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \left\{ f(x) \left[\frac{\alpha}{x - a} \int_a^x h'(t)g(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)g(t)dt \right] \right. \\ & \left. + g(x) \left[\frac{\alpha}{x - a} \int_a^x h'(t)f(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)f(t)dt \right] \right\} \\ & \left. + \left[\frac{\alpha}{x - a} \int_a^x h'(t)f(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)f(t)dt \right] \left[\frac{\alpha}{x - a} \int_a^x h'(t)g(t)dt + \frac{\beta}{b - x} \int_x^b h'(t)g(t)dt \right] \right| \\ & \leq (\alpha + \beta)^2 \left(\int_a^b |P(x, t)| \Delta t \right)^2, \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M_1 = \sup_{a < t < b} |f'(t)| < \infty \quad \text{and} \quad M_2 = \sup_{a < t < b} |g'(t)| < \infty.$$

Remark 3. *In the case of $h(t) = t$, $\alpha = x - a$ and $\beta = b - x$ in Corollary 7, we get the results given in [29] (for $h = 0$) and [31] (for $n = 1$).*

Corollary 8. *In the case of $\mathbb{T} = \mathbb{Z}$ in Theorem 2, we have*

$$\begin{aligned} & \left| \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\ & - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1)\Delta h(t) \right] \right. \\ & \left. + f(x) \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} g(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} g(t+1)\Delta h(t) \right] \right\} \right| \\ & \leq \frac{M_1 |g(x)| + M_2 |f(x)|}{2(\alpha + \beta)} \left[\sum_{t=a}^{b-1} |P(x, t)| \right] \end{aligned}$$

and

$$\begin{aligned}
& \left| f(x)g(x) \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right]^2 \right. \\
& - \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \left\{ f(x) \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} g(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} g(t+1)\Delta h(t) \right] \right. \\
& + g(x) \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1)\Delta h(t) \right] \left. \right\} \\
& + \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1)\Delta h(t) \right] \\
& \times \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} g(t+1)\Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} g(t+1)\Delta h(t) \right] \left. \right| \\
& \leq (\alpha + \beta)^2 \left(\sum_{t=a}^{b-1} |P(x, t)| \right)^2,
\end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x - 1, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b - 1 \end{cases}$$

and

$$M_1 = \sup_{a < t < b} |\Delta f(t)| < \infty \quad \text{and} \quad M_2 = \sup_{a < t < b} |\Delta g(t)| < \infty.$$

Remark 4. In the case of $h(t) = t$, $\alpha = x - a$ and $\beta = b - x$ in Corollary 8, we get the results given in [30, Theorem 1 and Theorem 2].

Corollary 9. In the case of $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 2, we have

$$\begin{aligned}
& \left| \frac{f(x)g(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \right. \\
& - \frac{1}{2(\alpha + \beta)} \left\{ g(x) \left[\frac{\alpha}{x - a} \int_a^x f(qt)D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt)D_q h(t) d_q t \right] \right. \\
& + f(x) \left[\frac{\alpha}{x - a} \int_a^x g(qt)D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b g(qt)D_q h(t) d_q t \right] \left. \right\} \left. \right| \\
& \leq \frac{M_1 |g(x)| + M_2 |f(x)|}{2(\alpha + \beta)} \left[\int_a^b |P(x, t)| d_q t \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left| f(x)g(x) \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right]^2 \right. \\
& - \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \left\{ f(x) \left[\frac{\alpha}{x - a} \int_a^x g(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b g(qt) D_q h(t) d_q t \right] \right. \\
& + g(x) \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right] \left. \right\} \\
& + \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right] \\
& \times \left[\frac{\alpha}{x - a} \int_a^x g(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b g(qt) D_q h(t) d_q t \right] \left. \right| \\
& \leq (\alpha + \beta)^2 \left(\int_a^b |P(x, t)| d_q t \right)^2,
\end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b \end{cases}$$

and

$$M_1 = \sup_{a < t < b} |(D_q f)(t)| < \infty \quad \text{and} \quad M_2 = \sup_{a < t < b} |(D_q g)(t)| < \infty.$$

3.3. New weighted perturbed Ostrowski type inequalities on time scales. In this subsection, four weighted perturbed Ostrowski type inequalities on time scales are established.

Theorem 3. Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f, h : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have

$$\begin{aligned}
& \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{f(b) - f(a)}{b - a} \left(\int_a^b P(x, t) \Delta t \right) \right. \\
& - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \left. \right| \\
& \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) \Delta t - \left(\frac{1}{b - a} \int_a^b P(x, t) \Delta t \right)^2 \right]^{\frac{1}{2}} \\
& \times \left[\frac{1}{b - a} \int_a^b (f^\Delta(t))^2 \Delta t - \left(\frac{f(b) - f(a)}{b - a} \right)^2 \right]^{\frac{1}{2}}, \quad f^\Delta \in L^2[a, b]; \\
(3.9) \quad & \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) \Delta t - \left(\frac{1}{b - a} \int_a^b P(x, t) \Delta t \right)^2 \right]^{\frac{1}{2}} \frac{\Gamma - \gamma}{2}, \quad \gamma \leq f^\Delta(x) \leq \Gamma, \quad x \in [a, b],
\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero,

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Proof. We have

$$\begin{aligned}
 & \frac{1}{b-a} \int_a^b P(x, t) f^\Delta(t) \Delta t - \left(\frac{1}{b-a} \int_a^b P(x, t) \Delta t \right) \left(\frac{1}{b-a} \int_a^b f^\Delta(t) \Delta t \right) \\
 (3.10) \quad &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (P(x, t) - P(x, s)) (f^\Delta(t) - f^\Delta(s)) \Delta t \Delta s.
 \end{aligned}$$

From (3.1), we also have

$$\begin{aligned}
 & \int_a^b P(x, t) f^\Delta(t) \Delta t \\
 &= \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\
 (3.11) \quad & - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right]
 \end{aligned}$$

and

$$(3.12) \quad \frac{1}{b-a} \int_a^b f^\Delta(t) \Delta t = \frac{f(b) - f(a)}{b-a}.$$

Using the Cauchy-Schwartz inequality, we may write

$$\begin{aligned}
 & \left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b (P(x, t) - P(x, s)) (f^\Delta(t) - f^\Delta(s)) \Delta t \Delta s \right| \\
 (3.13) \quad & \leq \left(\frac{1}{2(b-a)^2} \int_a^b \int_a^b (P(x, t) - P(x, s))^2 \Delta t \Delta s \right)^{\frac{1}{2}} \left(\frac{1}{2(b-a)^2} \int_a^b \int_a^b (f^\Delta(t) - f^\Delta(s))^2 \Delta t \Delta s \right)^{\frac{1}{2}}.
 \end{aligned}$$

However

$$(3.14) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (P(x, t) - P(x, s))^2 \Delta t \Delta s = \frac{1}{b-a} \int_a^b P^2(x, t) \Delta t - \left(\frac{1}{b-a} \int_a^b P(x, t) \Delta t \right)^2$$

and (see [26, inequality (3.3)])

$$\begin{aligned}
 & \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f^\Delta(t) - f^\Delta(s))^2 \Delta t \Delta s = \frac{1}{b-a} \int_a^b (f^\Delta(t))^2 \Delta t - \left(\frac{1}{b-a} \int_a^b f^\Delta(t) \Delta t \right)^2 \\
 &= \frac{1}{b-a} \int_a^b (f^\Delta(t))^2 \Delta t - \left(\frac{f(b) - f(a)}{b-a} \right)^2 \\
 (3.15) \quad & \leq \left[\frac{\Gamma - \gamma}{2} \right]^2, \text{ where } \gamma \leq f^\Delta(t) \leq \Gamma, \ t \in [a, b].
 \end{aligned}$$

Using (3.10)-(3.15), we can easily obtain the inequality (3.9). \square

Corollary 10. *In the case of $\mathbb{T} = \mathbb{R}$ in Theorem 3, we have*

$$\begin{aligned}
& \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{f(b) - f(a)}{b - a} \left(\int_a^b P(x, t) dt \right) \right. \\
& \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h'(t) f(t) dt + \frac{\beta}{b - x} \int_x^b h'(t) f(t) dt \right] \right| \\
& \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) dt - \left(\frac{1}{b - a} \int_a^b P(x, t) dt \right)^2 \right]^{\frac{1}{2}} \\
& \quad \times \left[\frac{1}{b - a} \int_a^b (f'(t))^2 dt - \left(\frac{f(b) - f(a)}{b - a} \right)^2 \right]^{\frac{1}{2}}, \quad f' \in L^2[a, b]; \\
& \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) dt - \left(\frac{1}{b - a} \int_a^b P(x, t) dt \right)^2 \right]^{\frac{1}{2}} \frac{\Gamma - \gamma}{2}, \quad \gamma \leq f'(x) \leq \Gamma, \quad x \in [a, b],
\end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Corollary 11. *In the case of $\mathbb{T} = \mathbb{Z}$ in Theorem 3, we have*

$$\begin{aligned}
& \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{f(b) - f(a)}{b - a} \left(\sum_{t=a}^{b-1} P(x, t) \right) \right. \\
& \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1) \Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1) \Delta h(t) \right] \right| \\
& \leq (b - a) \left[\frac{1}{b - a} \sum_{t=a}^{b-1} P^2(x, t) - \left(\frac{1}{b - a} \sum_{t=a}^{b-1} P(x, t) \right)^2 \right]^{\frac{1}{2}} \\
& \quad \times \left[\frac{1}{b - a} \sum_{t=a}^{b-1} (\Delta f(t))^2 - \left(\frac{f(b) - f(a)}{b - a} \right)^2 \right]^{\frac{1}{2}}; \\
& \leq (b - a) \left[\frac{1}{b - a} \sum_{t=a}^{b-1} P^2(x, t) - \left(\frac{1}{b - a} \sum_{t=a}^{b-1} P(x, t) \right)^2 \right]^{\frac{1}{2}} \frac{\Gamma - \gamma}{2}, \quad \gamma \leq \Delta f(t) \leq \Gamma, \quad t \in [a, b],
\end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x - 1, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b - 1. \end{cases}$$

Corollary 12. *In the case of $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 3, we have*

$$\begin{aligned}
& \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{f(b) - f(a)}{b - a} \left(\frac{1}{b - a} \int_a^b P(x, t) d_q t \right) \right. \\
& \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right] \right| \\
& \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) d_q t - \left(\frac{1}{b - a} \int_a^b P(x, t) d_q t \right)^2 \right]^{\frac{1}{2}} \\
& \quad \times \left[\frac{1}{b - a} \int_a^b (D_q f(t))^2 d_q t - \left(\frac{f(b) - f(a)}{b - a} \right)^2 \right]^{\frac{1}{2}}, \quad D_q f(t) \in L^2[a, b]; \\
& \leq (b - a) \left[\frac{1}{b - a} \int_a^b P^2(x, t) d_q t - \left(\frac{1}{b - a} \int_a^b P(x, t) d_q t \right)^2 \right]^{\frac{1}{2}} \frac{\Gamma - \gamma}{2}, \quad \gamma \leq D_q f(t) \leq \Gamma, \quad t \in [a, b],
\end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Theorem 4. *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f, h : [a, b] \rightarrow \mathbb{R}$ be differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \leq f^\Delta(x) \leq \Gamma$, $x \in [a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned}
& \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{\gamma + \Gamma}{2} \left(\int_a^b P(x, t) \Delta t \right) \right. \\
& \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] \right| \\
(3.16) \quad & \leq \frac{\Gamma - \gamma}{2} \left(\int_a^b |P(x, t)| \Delta t \right),
\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero,

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Proof. From (3.1), we may write

$$\begin{aligned}
& \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] \\
(3.17) \quad & = \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right] + \int_a^b P(x, t) f^\Delta(t) \Delta t.
\end{aligned}$$

Let $C = \frac{\gamma + \Gamma}{2}$. From (3.17), we get

$$\begin{aligned}
& \int_a^b P(x, t) (f^\Delta(t) - C) \Delta t \\
& = \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{\gamma + \Gamma}{2} \left(\int_a^b P(x, t) \Delta t \right) \\
(3.18) \quad & - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h^\Delta(t) f(\sigma(t)) \Delta t + \frac{\beta}{b - x} \int_x^b h^\Delta(t) f(\sigma(t)) \Delta t \right].
\end{aligned}$$

Using the properties of modulus, we get

$$(3.19) \quad \left| \int_a^b P(x, t) (f^\Delta(t) - C) \Delta t \right| \leq \frac{\Gamma - \gamma}{2} \left(\int_a^b |P(x, t)| \Delta t \right).$$

From (3.17)-(3.19), we can easily get (3.16). \square

Corollary 13. *In the case of $\mathbb{T} = \mathbb{R}$ in Theorem 4, we have*

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{\gamma + \Gamma}{2} \left(\int_a^b P(x, t) dt \right) \right. \\ & \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x h'(t) f(t) dt + \frac{\beta}{b - x} \int_x^b h'(t) f(t) dt \right] \right| \\ & \leq \frac{\Gamma - \gamma}{2} \left(\int_a^b |P(x, t)| dt \right), \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

Remark 5. *In the case of $h(t) = t$, $\alpha = x - a$ and $\beta = b - x$ in Corollary 13, we recapture the result given in [38, Corollary 1].*

Corollary 14. *In the case of $\mathbb{T} = \mathbb{Z}$ in Theorem 4, we have*

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{\gamma + \Gamma}{2} \left(\sum_{t=a}^{b-1} P(x, t) \right) \right. \\ & \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \sum_{t=a}^{x-1} f(t+1) \Delta h(t) + \frac{\beta}{b - x} \sum_{t=x}^{b-1} f(t+1) \Delta h(t) \right] \right| \\ & \leq \frac{\Gamma - \gamma}{2} \left(\sum_{t=a}^{b-1} |P(x, t)| \right), \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x - 1, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b - 1. \end{cases}$$

Corollary 15. *In the case of $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 4, we have*

$$\begin{aligned} & \left| \frac{f(x)}{\alpha + \beta} \left[\alpha \frac{h(x) - h(a)}{x - a} + \beta \frac{h(b) - h(x)}{b - x} \right] - \frac{\gamma + \Gamma}{2} \left(\int_a^b P(x, t) d_q t \right) \right. \\ & \quad \left. - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x - a} \int_a^x f(qt) D_q h(t) d_q t + \frac{\beta}{b - x} \int_x^b f(qt) D_q h(t) d_q t \right] \right| \\ & \leq \frac{\Gamma - \gamma}{2} \left(\int_a^b |P(x, t)| d_q t \right), \end{aligned}$$

where

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \left(\frac{h(t) - h(a)}{x - a} \right), & a \leq t < x, \\ \frac{-\beta}{\alpha + \beta} \left(\frac{h(b) - h(t)}{b - x} \right), & x \leq t < b. \end{cases}$$

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